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Rates of convergence for collocation with Jacobi polynomials for the airfoil equation

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Abstract

In this paper we consider for the classical airfoil equation a collocation method based on Jacobi polynomials. The error is estimated in suitable nonstandard Sobolev norms which are defined in such a way to respect the singularity structure of the exact solution. Furthermore, our numerical experiments underline our optimal theoretical error estimates.

Key words: Collocation method; Airfoil equations; Jacobi polynomials

0. Introduction

The classical airfoil equation

$$Sv(x) := \frac{1}{\pi}\int_{-1}^{1} \frac{v(y)}{y-x}\,\mathrm{d}y = f(x), \quad -1 < x < 1, \tag{1}$$

has for smooth f a solution of the form

$$v(x) = \rho(x)u(x),$$

where $\rho(x)$ is the weight function

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{2}$$

and $u(x)$ is "smooth". Here the integral is understood as a Cauchy principal value integral.

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The collocation method for the above equation to be considered here based on Chebyshev (Jacobi) polynomials is perhaps the easiest method of obtaining a numerical solution. It correctly represents the endpoint singularities of the exact solution, and yields faster than polynomial convergence if f is smooth.

We do not claim that the method is new. It has, for example, been suggested by Golberg [9,10] and others [1,3–6,11]. Our emphasis in the present work is on proving optimal rates of convergence in suitable Sobolev spaces (Theorems 1 and 6), by means of an analysis closely related to that used in [7,8,13]. Our analysis in Sections 1 and 2 is based on various discrete inner products. Numerical experiments which underline our theoretical results are given in Section 3.

In the following we list some well-known facts on Jacobi polynomials and explicit mapping properties of the integral operator S.

Let $\{p_j^{(\alpha,\beta)}(x)\}$, $j \geqslant 0$, be the Jacobi polynomials orthogonal with respect to the weight function $\rho(x)$, i.e., $\{p_n^{(1/2,-1/2)}(x)\}_0^\infty$, $\{p_n^{(-1/2,1/2)}(x)\}_0^\infty$ are Jacobi polynomials, normalized by the conditions

$$\int_{-1}^{1} (1-x)^{1/2}(1+x)^{-1/2} p_n^{(1/2,-1/2)}(x) p_m^{(1/2,-1/2)}(x)\, \mathrm{d}x = \delta_{mn},$$

$$\int_{-1}^{1} (1+x)^{1/2}(1-x)^{-1/2} p_n^{(-1/2,1/2)}(x) p_m^{(-1/2,1/2)}(x)\, \mathrm{d}x = \delta_{mn}.$$

$\{p_n^{(1/2,-1/2)}(x)\}_0^\infty$ is a complete orthogonormal basis for $L_{2,\rho}$, $\rho = (1-x)^{1/2}(1+x)^{-1/2}$, $\{p_n^{(-1/2,1/2)}(x)\}_0^\infty$ is a complete orthogonormal basis for $L_{2,1/\rho}$. Here α and β determined by

$$\alpha = \tfrac{1}{2} + m, \qquad \beta = -\tfrac{1}{2} + n, \tag{3}$$

and the integers m, n are related to the index ν of (1) by

$$\nu = -(\alpha + \beta) = -(m+n).$$

We will consider only nonnegative values of the index and, in order to have integrable solutions, we take α, $\beta > -1$. This restricts the index ν and the exponents α, β to the cases

$$\begin{array}{llll} (1) & \alpha = -\tfrac{1}{2}, & \beta = -\tfrac{1}{2}, & \nu = 1, \\ (2) & \alpha = \tfrac{1}{2}, & \beta = -\tfrac{1}{2}, & \nu = 0, \\ (3) & \alpha = -\tfrac{1}{2}, & \beta = \tfrac{1}{2}, & \nu = 0. \end{array} \tag{4}$$

When $\nu = 0$, the solution of (1) is unique whereas when $\nu = 1$, equation (1) usually has an eigensolution; to find a unique solution an additional side condition must be imposed like

$$l(u) = c, \tag{5}$$

where l is a given linear functional and c is a given constant.

Next, we introduce the notation for $\rho(x)$ from (2) with exponents as in (3):

$$S_\rho u(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\rho(y) u(y)}{y - x}\, \mathrm{d}y.$$

When $\nu = 0$, there holds (see [9])

$$S_\rho p_j^{(\alpha,\beta)}(x) := \frac{-1}{\sin(\pi\alpha)} p_j^{(-\alpha,-\beta)}, \quad j = 0, 1, \ldots,$$

i.e., for $\rho = (1-x)^{1/2}(1+x)^{-1/2}$,

$$S_\rho p_j^{(1/2,-1/2)}(x) = -p_j^{(-1/2,1/2)}(x), \quad j = 0, 1, \ldots, \tag{6}$$

and for $\rho = (1-x)^{-1/2}(1+x)^{1/2}$,

$$S_\rho p_j^{(-1/2,1/2)}(x) = p_j^{(1/2,-1/2)}(x), \quad j = 0, 1, \ldots.$$

When $\nu = 1$, there holds (see [9])

$$S_\rho p_j^{(\alpha,\beta)}(x) = \begin{cases} -\dfrac{1}{\sin(\pi\alpha)} p_{j-1}^{(-\alpha,-\beta)}(x), & j = 1, 2, \ldots, \\ 0, & j = 0, \end{cases}$$

i.e., for $\rho = (1-x)^{-1/2}(1+x)^{-1/2}$,

$$S_\rho p_j^{(-1/2,-1/2)}(x) = \begin{cases} p_{j-1}^{(1/2,1/2)}(x), & j = 1, 2, \ldots, \\ 0, & j = 0. \end{cases} \tag{7}$$

The case (3) in (4) does not inherit any substantial differences to case (2), so we consider only the cases (1) and (2).

1. Method A

In order to make the paper more readable we use for the Method A, i.e., $\alpha = \beta = -\frac{1}{2}$, $\nu = 1$, the common notation for Chebyshev polynomials. There holds from (7),

$$S_\rho T_j = U_{j-1}, \quad j = 1, 2, \ldots, \qquad S_\rho T_0 = 0,$$

where $T_j = p_j^{(-1/2,-1/2)}$ are the Chebyshev polynomials of the first kind and $U_j = p_j^{(1/2,1/2)}$ are the Chebyshev polynomials of the second kind normalized by

$$\int_{-1}^{1} (1-x^2)^{-1/2} T_j(x) T_k(x)\, \mathrm{d}x = \begin{cases} \delta_{jk} \cdot \frac{1}{2}\pi, & k \neq 0, \\ \delta_{jk} \cdot \pi, & k = 0, \end{cases}$$

and

$$\int_{-1}^{1} (1-x^2)^{1/2} U_j(x) U_k(x)\, \mathrm{d}x = \delta_{jk} \cdot \tfrac{1}{2}\pi.$$

We approximate the solution ν of (1) by $\rho(x) = 1/\sqrt{1-x^2}$ times a polynomial u_n of degree $\leqslant n-1$:

$$v_n(x) = \rho(x) u_n(x) = \rho(x) \left(\sum_{k=0}^{n-1} a_k T_k(x) \right), \quad a_k \in \mathbb{R}, \; k = 0, \ldots, n-1. \tag{8}$$

We choose for the collocation points

$$\tilde{x}_j = \cos\frac{j}{n}\pi, \quad 1 \leqslant j \leqslant n-1,$$

the $n-1$ zeros of $U_{n-1}(x)$. Thus our collocation scheme is: find $u_n \in \mathbb{P}_{n-1}$, such that

$$S_\rho u_n(\tilde{x}_j) = f(\tilde{x}_j), \quad 1 \leqslant j \leqslant n-1. \tag{9}$$

Here $\mathbb{P}_{n-1}$ denotes the space of polynomials of degree $\leqslant n-1$ on $(-1, 1)$. Now, with the aid of (7), we have

$$S_\rho u_n(x) = \sum_{k=1}^{n-1} a_k U_{k-1}(x). \tag{10}$$

On the other hand, the choice $c = 0$ and

$$l(u_n) = \int_{-1}^{1} \rho(x) u_n(x)\, \mathrm{d}x \overset{!}{=} 0 \tag{11}$$

in (5) leads to the constraint

$$a_0 = 0.$$

Thus equations (9) and (11) represent a set of linear equations for $a_0, a_1, \ldots, a_{n-1}$ with easily computed matrix elements. One may even obtain explicit expressions for $a_1, \ldots, a_{n-1}$ by exploiting the discrete orthogonality of Chebyshev polynomials of the second kind. Let us define an inner product incorporating the weight $1/\rho$ by

$$\langle\langle u, w \rangle\rangle = \int_{-1}^{1} u(x) w(x) \frac{1}{\rho(x)} \mathrm{d}x, \tag{12}$$

and a corresponding discrete inner product (obtained by using the Gaussian quadrature rule for the weight $(1-x)^{1/2}(1+x)^{1/2}$ where the nodes x_k are the zeros of $U_n(x)$)

$$\langle\langle u, w \rangle\rangle_n = \sum_{k=1}^{n} \omega_k u(x_k) w(x_k).$$

We have

$$\omega_k = \frac{\pi}{n+1} \sin^2\left(\frac{k}{n+1}\right)\pi, \quad k = 1, \ldots, n.$$

Then, as the n-point Gaussian quadrature rule is exact for all polynomials of degree $\leqslant 2n-1$, we have, for $j, k = 0, \ldots, n-1$, the discrete orthogonality property

$$\langle\langle U_j, U_k \rangle\rangle_n = \langle\langle U_j, U_k \rangle\rangle = \delta_{jk} \cdot \tfrac{1}{2}\pi. \tag{13}$$

It then follows from (9), (10) and (13) that the coefficients in (8) have the explicit form

$$a_j = \frac{2}{\pi} \langle\langle f, U_{j-1} \rangle\rangle_{n-1}, \quad j = 1, \ldots, n-1, \qquad a_0 = 0.$$

In particular, if $f \in \mathbb{P}_{n-1}$, then Method A yields the exact solution.

In order to present the convergence results, we begin by defining some norms by which the error may be described.

Let u have a Chebyshev polynomial expansion

$$u(x) = \sum_{k=0}^{\infty} \tilde{u}(k) T_k(x), \quad \text{with} \begin{cases} \tilde{u}(k) = \dfrac{2}{\pi}\langle u, T_k \rangle, & k \in \mathbb{Z}_+, \\ \tilde{u}(0) = \dfrac{1}{\pi}\langle u, T_0 \rangle, \end{cases} \tag{14}$$

and $\tilde{u}(0) = 0$ by (11) where

$$\langle u, w \rangle = \int_{-1}^{1} \rho(x) u(x) w(x) \, \mathrm{d}x.$$

Then we define the space $\tilde{H}_\rho^s$, $s \in \mathbb{R}$, to be the closure of all polynomials with respect to the norm

$$\| u \|_{\tilde{H}_\rho^s}^2 = \sum_{k=1}^{\infty} | k |^{2s} | \tilde{u}(k) |^2 + | \tilde{u}(0) |^2.$$

Note $\| u \|_{\tilde{H}_\rho^0}^2 = \langle u, u \rangle = \| u \|_{L_{2,\rho}}^2$.

Theorem 1. *Let $u \in \tilde{H}_\rho^t$ solve (1) with $u = v/\rho$ and $\rho(x) = (1 - x^2)^{-1/2}$. Then for any $n \in \mathbb{Z}_+$ there exists a solution $u_n \in \mathbb{P}_{n-1}$ of (9). Furthermore, if $t > s$ and $t > \frac{1}{2}$, then u_n satisfies*

$$\| u_n - u \|_{\tilde{H}_\rho^s} \leqslant \begin{cases} C n^{-(t-s)} \| u \|_{\tilde{H}_\rho^t}, & s \geqslant 0, \\ C n^{-t} \| u \|_{\tilde{H}_\rho^t}, & s < 0, \end{cases}$$

with C a constant independent of n.

Remark 2. In [9] Golberg shows the convergence of the above methods with a different analysis, but there error estimates are not given.

Theorem 3. *The operator $S_\rho : \tilde{H}_\rho^s \backslash \{T_0\} \to \hat{H}_{1/\rho}^s$ is continuous with continuous inverse where the space $\hat{H}_{1/\rho}^s$, $s \in \mathbb{R}$, is the closure of all polynomials with respect to the norm*

$$\| f \|_{\hat{H}_{1/\rho}^s} = \left(\sum_{k=0}^{\infty} | k + 1 |^{2s} | \hat{f}(k) |^2 \right)^{1/2},$$

where $\hat{f}(k) = 2/\pi \langle\langle f, U_k \rangle\rangle$, $k \in \mathbb{Z}_+$, with (12).

Proof of Theorem 1. The approximation scheme (9) is equivalent to

$$\langle\langle S_\rho u_n, U_j \rangle\rangle_{n-1} = \langle\langle S_\rho u, U_j \rangle\rangle_{n-1}, \quad j = 0, \ldots, n-2. \tag{15}$$

Now, from (7) and (14) we have

$$S_\rho u(x) = \sum_{k=1}^{\infty} \tilde{u}(k) U_{k-1}(x).$$

Thus,

$$\langle\langle S_\rho u, U_j\rangle\rangle_{n-1} = \sum_{k=1}^{\infty} \tilde{u}(k)\langle\langle U_{k-1}, U_j\rangle\rangle_{n-1}, \quad j=0,\dots,n-2,$$

or

$$\langle\langle S_\rho u, U_j\rangle\rangle_{n-1} = \tfrac{1}{2}\pi\tilde{u}(j+1) + \sum_{k=2n-1-j}^{\infty} \tilde{u}(k)\langle\langle U_{k-1}, U_j\rangle\rangle_{n-1}, \quad 0 \leqslant j \leqslant n-2.$$

Similarly, there holds

$$\langle\langle S_\rho u_n, U_j\rangle\rangle_{n-1} = \tfrac{1}{2}\pi\tilde{u}_n(j+1), \quad 0 \leqslant j \leqslant n-2.$$

Using (15), we obtain

$$\tilde{u}_n(j+1) = \tilde{u}(j+1) + \frac{2}{\pi}\sum_{k=2n-j-1}^{\infty} \tilde{u}(k)\langle\langle U_{k-1}, U_j\rangle\rangle_{n-1}, \quad 0 \leqslant j \leqslant n-2. \tag{16}$$

To estimate the error $u_n - u$ in the $\tilde{H}_\rho^s$ norm we proceed as follows. Since $\tilde{u}_n(k) = 0$ for $k \geqslant n$, we have

$$\| u_n - u \|_{\tilde{H}_\rho^s}^2 = \sum_{k=1}^{n-1} k^{2s} |\tilde{u}_n(k) - \tilde{u}(k)|^2 + \sum_{k=n}^{\infty} k^{2s} |\tilde{u}(k)|^2. \tag{17}$$

Using that Chebyshev polynomials satisfy [8] for $j = 0,\dots,n-2$,

$$\langle\langle U_k, U_j\rangle\rangle_{n-1} = \begin{cases} -\tfrac{1}{2}\pi, & k = 2nm-2-j, \quad m = 1, 2, \dots, \\ \tfrac{1}{2}\pi, & k = 2nm+j, \quad m = 0, 1, \dots, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

we estimate the second term in (17):

$$\sum_{k=n}^{\infty} k^{2s} |\tilde{u}(k)|^2 = \sum_{k=n}^{\infty} k^{2s-2t} k^{2t} |\tilde{u}(k)|^2 \leqslant n^{2s-2t} \| u \|_{\tilde{H}_\rho^t}^2,$$

provided $s < t$.

Finally, we consider the first term in (17). With (16) we have

$$\begin{aligned} &\sum_{j=0}^{n-2} (j+1)^{2s} |\tilde{u}_n(j+1) - \tilde{u}(j+1)|^2 \\ &\quad = \sum_{j=0}^{n-2} (j+1)^{2s} \left(\frac{2}{\pi} \sum_{k=2n-j-1}^{\infty} \tilde{u}(k)\langle\langle U_{k-1}, U_j\rangle\rangle_{n-1} \right)^2 \\ &\quad = \sum_{j=0}^{n-2} (j+1)^{2s} \left(\frac{2}{\pi} \sum_{k=2n-j-2}^{\infty} \tilde{u}(k+1)\langle\langle U_k, U_j\rangle\rangle_{n-1} \right)^2 \\ &\quad = \sum_{j=1}^{n-2} (j+1)^{2s} \left(\sum_{m=1}^{\infty} \tilde{u}(2nm+j+1) - \tilde{u}(2nm-j-1) \right)^2, \quad \text{with (18).} \end{aligned}$$

If $s \geqslant 0$, we have $(j+1)^{2s} \leqslant n^{2s}$, whereas if $s<0$, we have $(j+1)^{2s} \leqslant 1$. Thus for $s \geqslant 0$ we obtain for $t > \frac{1}{2}$, using the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\sum_{j=0}^{n-2} (j+1)^{2s} |\tilde{u}_n(j+1) - \tilde{u}(j+1)|^2 \\
&\quad \leqslant n^{2s} \sum_{j=0}^{n-2} \left| \sum_{m=1}^{\infty} \tilde{u}(2nm+j+1) - \sum_{m=1}^{\infty} \tilde{u}(2nm-j-1) \right|^2 \\
&\quad \leqslant n^{2s} \sum_{j=0}^{n-2} \sum_{m=1}^{\infty} (2nm+j+1)^{-2t} \sum_{m=1}^{\infty} (2nm+j+1)^{2t} |\tilde{u}(2nm+j+1)|^2 \\
&\qquad + n^{2s} \sum_{j=0}^{n-2} \sum_{m=1}^{\infty} (2nm-j-1)^{-2t} \sum_{m=1}^{\infty} (2nm-j-1)^{2t} |\tilde{u}(2nm-j-1)|^2 \\
&\quad \leqslant C n^{2s-2t} \sum_{j=0}^{n-2} \left[\sum_{m=1}^{\infty} (2nm+j+1)^{2t} |\tilde{u}(2nm+j+1)|^2 \right. \\
&\qquad\qquad \left. + \sum_{m=1}^{\infty} (2nm-j-1)^{2t} |\tilde{u}(2nm-j-1)|^2 \right] \\
&\quad \leqslant C n^{2s-2t} \sum_{k=n}^{\infty} k^{2t} |\tilde{u}(k)|^2 \leqslant C n^{2s-2t} \| u \|^2_{\tilde{H}^t_\rho},
\end{aligned}
$$

using $\sum_{m=1}^{\infty}(2nm-j-1)^{-2t} \leqslant \sum_{m=1}^{\infty}(nm)^{-2t} \leqslant Cn^{-2t}$. For $s<0$ we estimate correspondingly, provided $t > \frac{1}{2}$. □

Proof of Theorem 3. Let $f \in \hat{H}^s_{1/\rho}$ and $S_\rho u = f$. Then $f(x) = \sum_{k=0}^{\infty} \hat{f}(k) U_k(x)$, where

$$\| f \|^2_{\hat{H}^s_{1/\rho}} = \sum_{k=0}^{\infty} |k+1|^{2s} |\hat{f}(k)|^2.$$

But due to (7) we have, with $u = \sum_{k=0}^{\infty} \tilde{u}(k) T_k$ where $\tilde{u}(0) = 0$ since $u \in \tilde{H}^s_\rho \setminus \{T_0\}$,

$$S_\rho u = \sum_{k=0}^{\infty} \tilde{u}(k) S_\rho T_k = \sum_{k=1}^{\infty} \tilde{u}(k) U_{k-1} = \sum_{k=1}^{\infty} \hat{f}(k-1) U_{k-1},$$

yielding

$$\tilde{u}(k) = \hat{f}(k-1), \quad \text{for } k \geqslant 1.$$

Hence,

$$\| f \|^2_{\hat{H}^s_{1/\rho}} = \sum_{k=0}^{\infty} |k+1|^{2s} |\tilde{u}(k+1)|^2 = \sum_{l=1}^{\infty} |l|^{2s} |\tilde{u}(l)|^2 = \| u \|^2_{\tilde{H}^s_\rho} - |\tilde{u}(0)|^2,$$

from which the assertion follows. □

Remark 4. For $\tau = 0$ we have $\hat{H}^{\tau}_{1/\rho} = L_{2,1/\rho}$, $\tilde{H}^{\tau}_{\rho} = L_{2,\rho}$, where

$$\| u \|^2_{L_{2,\rho}} = \int_{-1}^{1} \rho u^2 \, dx \quad \text{and} \quad \| f \|^2_{L_{2,1/\rho}} = \int_{-1}^{1} \frac{1}{\rho} f^2 \, dx.$$

The continuity of the mapping property $S_\rho : L_{2,\rho} \to L_{2,1/\rho}$ is well known [9].

Remark 5. Due to Theorems 1 and 3 we have for $f \in \hat{H}^t_{1/\rho}$ with $t > s$ and $t > \frac{1}{2}$ the estimate

$$\| u_n - u \|_{\tilde{H}^s_\rho} \leqslant \begin{cases} Cn^{-(t-s)} \| f \|_{\hat{H}^t_{1/\rho}}, & s \geqslant 0, \\ Cn^{-t} \| f \|_{\hat{H}^t_{1/\rho}}, & s < 0, \end{cases}$$

with the notation of Theorem 1.

2. Method B

For $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ we look for solutions of (1) of the form

$$v(x) = \sqrt{\frac{1-x}{1+x}} \, u(x).$$

Due to property (6) it is natural to approximate the solution v of (1) by $\rho(x) = (1-x)^{1/2}(1+x)^{-1/2}$ times a polynomial u_n of degree $\leqslant n-1$:

$$v_n(x) = \rho(x) \sum_{k=0}^{n-1} a_k p_k^{(1/2,-1/2)}(x), \quad a_k \in \mathbb{R}.$$

We choose the collocation points

$$x_k = \cos \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, \ldots, n, \tag{19}$$

which are the distinct zeros of $p_n^{(-1/2,1/2)}(x)$. Thus our collocation scheme is: find $u_n \in \mathbb{P}_{n-1}$ such that

$$S_\rho u_n(x_k) = f(x_k), \quad 1 \leqslant k \leqslant n. \tag{20}$$

Here,

$$\begin{aligned} p_n^{(-1/2,1/2)}(x) &= \frac{1}{\sqrt{\pi}} \frac{\cos\left(\left(n+\frac{1}{2}\right)\cos^{-1}x\right)}{\cos\left(\frac{1}{2}\cos^{-1}x\right)}, \quad n = 0, 1, \ldots, \\ p_n^{(1/2,-1/2)}(x) &= \frac{1}{\sqrt{\pi}} \frac{\sin\left(\left(n+\frac{1}{2}\right)\cos^{-1}x\right)}{\sin\left(\frac{1}{2}\cos^{-1}x\right)}, \quad n = 0, 1, \ldots. \end{aligned} \tag{21}$$

Now, with the aid of (6),

$$S_\rho u_n(x) = -\sum_{k=0}^{n-1} a_k p_k^{(-1/2,1/2)}(x). \tag{22}$$

Note that (20) represents a set of linear equations for $a_0,\dots,a_{n-1}$ with easily computed matrix elements. Again, one may even obtain explicit expressions for $a_0,\dots,a_{n-1}$ by exploiting the discrete orthogonality of the Jacobi polynomials $\{p_k^{(-1/2,1/2)}(x)\}_{n=0}^{\infty}$. There holds the following discrete inner product (obtained by using the Gauss–Jacobi quadrature rule based on the weight $(1/\rho)(x)=(1-x)^{-1/2}(1+x)^{1/2}$ which uses the zeros (19) of $p_n^{(-1/2,1/2)}(x)$ as nodes and the weights ω_k (see [12])):

$$\langle u, w\rangle_n=\sum_{k=1}^{n}\omega_k u(x_k)w(x_k). \tag{23}$$

Then there holds

$$\begin{aligned}\langle p_j^{(-1/2,1/2)}, p_k^{(-1/2,1/2)}\rangle_n &= \int_{-1}^{1} p_j^{(-1/2,1/2)}(x)p_k^{(-1/2,1/2)}(x)\frac{1}{\rho}(x)\,\mathrm{d}x\\ &=\delta_{jk},\quad j,k\leqslant n-1.\end{aligned}$$

Hence, from (22) follows

$$a_k=-\langle f, p_k^{(-1/2,1/2)}\rangle_n,\quad k=0,\dots,n-1.$$

With

$$\begin{aligned}\|u\|_{\bar{H}_\rho^s}^2 &= \sum_{k=1}^{\infty}|k|^{2s}|\hat{u}(k)|^2+|\hat{u}(0)|^2,\\ u(x) &= \sum_{k=0}^{\infty}\hat{u}(k)p_k^{(1/2,-1/2)}(x),\\ \hat{u}(k) &= \int_{-1}^{1}u(x)p_k^{(1/2,-1/2)}(x)\rho(x)\,\mathrm{d}x,\end{aligned} \tag{24}$$

we have the following theorem.

Theorem 6. *Let $u\in\bar{H}_\rho^t$ solve (1) with $u=v/\rho$ and $\rho(x)=(1-x)^{1/2}(1+x)^{-1/2}$. Then for any $n\in\mathbb{Z}_+$ there exists a solution $u_n\in\mathbb{P}_{n-1}$ of (20). Furthermore, if $t>s$ and $t>\frac{1}{2}$, then u_n satisfies*

$$\|u_n-u\|_{\bar{H}_\rho^s}\leqslant\begin{cases}Cn^{-(t-s)}\|u\|_{\bar{H}_\rho^t}, & s\geqslant 0,\\ Cn^{-t}\|u\|_{\bar{H}_\rho^t}, & s<0,\end{cases} \tag{25}$$

with C a constant independent of n.

Proof. The approximation scheme (20) is equivalent to

$$\langle S_\rho u_n, p_j^{(-1/2,1/2)}\rangle_n=\langle S_\rho u, p_j^{(-1/2,1/2)}\rangle_n,\quad j=0,\dots,n-1. \tag{26}$$

Again one needs a discrete orthogonality property like (18) but now for the orthogonal system $\{p_k^{(-1/2,1/2)}(x)\}_{k=0}^{\infty}$ with respect to the discrete inner product in (23). This discrete orthogonality property can be obtained by using (21). One obtains for $j=0,\dots,n-1$,

$$\langle p_k^{(-1/2,1/2)}, p_j^{(-1/2,1/2)}\rangle_n = \begin{cases} -(-1)^m, & k=(2n+1)m-1-j, \quad m=1,2,\dots, \\ (-1)^m, & k=(2n+1)m+j, \quad m=0,1,\dots, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

Now from (6) and (24) with $v=\rho u$, $\rho=\sqrt{(1-x)/(1+x)}$ we have

$$S_\rho u(x) = -\sum_{k=0}^{\infty} \hat{u}(k) p_k^{(-1/2,1/2)}(x).$$

Thus we can compute with (27) for $j=0,\dots,n-1$,

$$\begin{aligned} &\langle S_\rho u, p_j^{(-1/2,1/2)}\rangle_n \\ &= -\sum_{k=0}^{\infty} \hat{u}(k)\langle p_k^{(-1/2,1/2)}, p_j^{(-1/2,1/2)}\rangle_n \\ &= -\sum_{m=0}^{\infty}\sum_{r=0}^{2n} \hat{u}(r+(2n+1)m)\langle p_{(2n+1)m+r}^{(-1/2,1/2)}, p_j^{(-1/2,1/2)}\rangle_n \\ &= -\sum_{m=0}^{\infty} (-1)^m \hat{u}(j+(2n+1)m) + \sum_{m=1}^{\infty} (-1)^m \hat{u}(-j-1+(2n+1)m) \\ &= -\hat{u}(j) - \sum_{m=1}^{\infty} (-1)^m \hat{u}(j+(2n+1)m) + \sum_{m=1}^{\infty} (-1)^m \hat{u}(-j-1+(2n+1)m). \end{aligned} \tag{28}$$

On the other hand, there holds for $j=0,\dots,n-1$,

$$\langle S_\rho u_n, p_j^{(-1/2,1/2)}\rangle_n = -\hat{u}_n(j). \tag{29}$$

Thus one obtains from (26) with (28) and (29) for $j=0,\dots,n-1$,

$$\hat{u}(j) - \hat{u}_n(j) = -\sum_{m=1}^{\infty} (-1)^m \hat{u}(j+(2n+1)m) + \sum_{m=1}^{\infty} (-1)^m \hat{u}(-j-1+(2n+1)m).$$

Next we estimate the norm in (24):

$$\| u_n - u \|_{H_\rho^s}^2 = \sum_{j=0}^{n-1} j^{2s} |\hat{u}_n(j) - \hat{u}(j)|^2 + \sum_{j=n}^{\infty} j^{2s} |\hat{u}(j)|^2. \tag{30}$$

First we estimate with the second term in (30) for $s<t$,

$$\sum_{j=n}^{\infty} j^{2s} |\hat{u}(j)|^2 = \sum_{j=n}^{\infty} j^{2s-2t} j^{2t} |\hat{u}(j)|^2 \leqslant n^{2s-2t} \| u \|_{H_\rho^t}^2.$$

Next we consider with the first term in (30) for $s \geqslant 0$ and $t > s$,

$$\begin{aligned}
&\sum_{j=0}^{n-1} j^{2s} \left| - \sum_{m=1}^{\infty} (-1)^m \hat{u}(j + (2n+1)m) + \sum_{m=1}^{\infty} (-1)^m \hat{u}(-j-1+(2n+1)m) \right|^2 \\
&\quad \leqslant n^{2s} \sum_{j=0}^{n-1} \left(\sum_{m=1}^{\infty} (j+(2n+1)m)^{-2t} \sum_{m=1}^{\infty} (j+(2n+1)m)^{2t} \left| \hat{u}(j+(2n+1)m) \right|^2 \right. \\
&\qquad + \sum_{m=1}^{\infty} (-j-1+(2n+1)m)^{-2t} \\
&\qquad \left. \times \sum_{m=1}^{\infty} (-j-1+(2n+1)m)^{2t} \left| \hat{u}(-j-1+(2n+1)m) \right|^2 \right) \\
&\quad \leqslant C n^{2s-2t} \sum_{k=n+1}^{\infty} k^{2t} \left| \hat{u}(k) \right|^2 \leqslant C n^{2s-2t} \| u \|^2_{\tilde{H}^t_\rho}.
\end{aligned}$$

For $s < 0$ we use $j^{2s} \leqslant 1$ and obtain analogously the desired estimate (25), provided $t > \frac{1}{2}$. □

3. Numerical results

The following numerical results were performed on the CDC 855 at the University of Hannover.

Table 1
Weighted L^2-error for collocation scheme (9)

$(1-x^2)^{3/2}$			e^{-x}		
n	$\|e\|_{\tilde{H}^0_\rho}$	α_n	n	$\|e\|_{\tilde{H}^0_\rho}$	α_n
6	$0.525 \cdot 10^{-2}$		3	0.170	
		4.56			6.31
11	$0.332 \cdot 10^{-3}$		4	$0.277 \cdot 10^{-1}$	
		5.15			9.38
16	$0.481 \cdot 10^{-4}$		5	$0.342 \cdot 10^{-2}$	
		4.41			12.68
21	$0.145 \cdot 10^{-4}$		6	$0.339 \cdot 10^{-3}$	
		4.85			16.16
26	$0.515 \cdot 10^{-5}$		7	$0.281 \cdot 10^{-4}$	
		4.46			19.80
31	$0.235 \cdot 10^{-5}$		8	$0.199 \cdot 10^{-5}$	
		4.79			23.57
36	$0.115 \cdot 10^{-5}$		9	$0.124 \cdot 10^{-6}$	
		4.51			27.46
41	$0.638 \cdot 10^{-6}$		10	$0.688 \cdot 10^{-8}$	
		4.78			31.46
46	$0.368 \cdot 10^{-6}$		11	$0.343 \cdot 10^{-9}$	
		4.57			35.55
51	$0.230 \cdot 10^{-6}$		12	$0.156 \cdot 10^{-10}$	
α^*		4.5			

In Table 1 we give the weighted L^2-error for the collocation approximation of (1) and present the associated experimental convergence rates α_n.

For f listed in Table 1 we approximate the unknown solution v of (1) by taking $\tilde{v} = \rho u_{200}^*$ with $\rho(x) = 1/\sqrt{1-x^2}$ as proposed in Method A. Here u_{200}^* solves the collocation scheme (9) for $n = 200$. Thus instead of the actual error $u - u_n$, where $u = v/\rho$ and u_n solves (9) and satisfies the additional constraint (11), we list in Table 1 the weighted L^2-error for $e = u_{200}^* - u_n$, i.e.,

$$\| e \|_{\tilde{H}_\rho^0} = \left(\int_{-1}^{1} (1-x^2)^{-1/2} \, | u_{200}^* - u_n |^2 \, \mathrm{d}x \right)^{1/2}. \tag{31}$$

In evaluating (31) we use the respective Chebyshev polynomial expansions (8) for u_{200}^* and u_n and perform the integration by the 200-point Gauss–Chebyshev quadrature rule. Note that $(1-x^2)^{3/2} \in \hat{H}_{1/\rho}^{9/2-\epsilon}$ for any $\epsilon > 0$, and $\mathrm{e}^{-x} \in \hat{H}_{1/\rho}^{s}$ for any s, which can be seen by computing the Chebyshev coefficients [7]. Therefore Theorem 1 together with Theorem 3 gives the theoretical convergence rate α^* listed in Table 1 (cf. Remark 5). For $f = \mathrm{e}^{-x}$ Table 1 shows faster than polynomial convergence.

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